CU-TP-552
IASSNS-HEP-92/13

Classical Solutions in Quantum Field Theories

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This work was supported in part by the US Department of Energy, by the Monell Foundation, and by NASA under grant NAGW-2381

1. INTRODUCTION

Realistic quantum field theories are difficult to solve because they are governed by nonlinear operator equations. In the usual perturbative treatment one begins with the solution to the linearized (free-field) version of the theory and then incorporates the effects of the interactions as a power series expansion in some small coupling. A complementary approach, in which the nonlinearity of the system is retained at all stages in the calculation, is based on an expansion about solutions of classical field equations; the higher order corrections in this approach are obtained as a power series in the same small coupling. This approach has led to new insight into the properties of quantum field theories. In this article I will give an overview of these methods and describe some of the most important results.

In many cases one finds that the classical field equations have solutions which suggest a particle interpretation. They are localized, with their energy density concentrated within a fairly well defined region of space. Outside this region, the fields rapidly approach their vacuum values. These solutions are stable and maintain their form as time goes on. Finally, they can be boosted to give linearly moving solutions. These carry linear momentum, and display the proper relationship between mass, momentum, and energy.

The existence of these objects, known as solitons,[1] depends crucially on the nonlinear nature of the field equations. This is reflected in their nonanalytic behavior as the coupling constants of the theory approach zero. In particular, the soliton mass typically diverges in this limit, behaving as an inverse of a coupling constant.

The quantum theory contains particle states corresponding to these classical objects. These states are not accessible to ordinary perturbation theory. Nevertheless, they are most easily studied in the weak coupling limit, where the Compton

1 This usage differs from that in other fields, where the term soliton is applied only to localized solutions which maintain their form even after scattering. Except in the sine-Gordon theory, none of the classical solutions encountered in high energy physics are solitons in this more restricted sense.

wavelength of the massive soliton is much smaller than the spatial extent of the classical solution. This makes it possible to localize the soliton, and allows one to use the classical field configuration as the basis for a description of the internal structure of the particle.

How do we know that a particular theory has soliton solutions? One way, of course, is to actually obtain analytic solutions to the field equations. However, this turns out to be feasible only in a very few cases, most of them idealized models rather than phenomenologically relevant theories. In any case, simply displaying the solution does not explain the physical basis for its existence.

In many theories with spontaneous symmetry breaking, topological arguments can be used to establish the existence of solutions. In these topological solitons, the fields approach different degenerate vacua as one approaches spatial infinity in different directions. These vacua are chosen in such a fashion that they cannot be continuously deformed to a single vacuum. This guarantees the stability of the soliton, and gives rise to a new type of conserved quantum number, known as the topological charge.

A second class of solutions, nontopological solitons, are also stabilized by a conserved charge carried by the soliton. However, in this case the charge is of the same kind as that carried by the elementary particles of the theory. For stable solitons to exist, their mass to charge ratio must be small enough to prevent decay by emission of these elementary charged particles.

For the solutions discussed above, the quantum interpretation is a straightforward extension of the classical meaning. This is not the case for instantons, which are solutions of the Euclidean field equations; i.e., the equations obtained by continuing to imaginary time. These are associated with quantum mechanical tunneling. This connection can be motivated by recalling the WKB treatment of one-dimensional barrier penetration, where the exponent in the tunneling amplitude is of the form of an action $\int p\, dq$, but with the "wrong" sign in the equation relating the momentum to the energy. This nonstandard sign is just what one would obtain by doing mechanics with imaginary time. The one-dimensional WKB approximation can be extended to systems with many degrees of freedom, where the barrier is in a multidimensional configuration space. The tunneling amplitude is then obtained by considering all possible paths through the barrier, and finding the one for which the one-dimensional tunneling probability is the greatest. In a field theory, each point along this path corresponds to a specification of a field configuration over all of three-dimensional space. The path, being a sequence of such configurations, can itself be viewed as a configuration in a four-dimensional space. This configuration turns out to be given by a solution of the Euclidean field equations, namely the instanton. Thus, the instanton is not an object existing in real space, but rather a device for calculating a quantum mechanical amplitude.

One application of such solutions is to the decay of a classically stable, but quantum mechanically metastable, state. This situation arises in some cosmological scenarios, since the early universe could have been for a time in a "false vacuum" state, corresponding to a local, but not global, minimum of the field potential $V(\phi)$. Such a state would have decayed by the nucleation of bubbles of the true vacuum, with the nucleation occuring (at low temperatures) through quantum mechanical tunneling. The Euclidean solution associated with this process is often referred to as a bounce solution.

In nonrelativistic quantum mechanics tunneling plays a role in the treatment of systems which have two or more degenerate classical minima separated by potential energy barriers. The standard example is that of a particle in a double well potential where, because of tunneling, the ground state is a linear combination of the ground states of the two wells, and the splitting of the ground state from the first excited state is obtained from the WKB tunneling amplitude. Similar phenomena can also occur in field theory. In particular, non-Abelian gauge theories can be viewed as having multiple vacua separated by finite energy barriers. Tunneling between these vacua, described by the Yang-Mills instanton solution, has a number of important consequences for both QCD and the standard electroweak theory.

Of course, there is more than one way to get to the other side of a potential energy barrier. At high temperature, a system may have enough kinetic energy that it can go over the barrier without the need for quantum tunneling. In a multidimensional configuration space this process will occur most readily across the point where the barrier is lowest. The saddle point, i.e. the high point on the lowest path over the barrier, is a stationary point of the potential energy and thus a static, although unstable, solution of the field equations. Such solutions have come to be known as sphalerons.

To illustrate some of the features of solitons and the issues involved in going from the classical solution to the quantum theory, I begin in Sec. 2 by discussing the kink, a soliton in one space dimension. Although not of direct physical interest (except as a model for some cosmological domain walls), this example has the advantage of being simple enough that much of the analysis can be done explicitly. I then go on in Sec. 3 to discuss more complex solitons, both topological and nontopological. As examples of the former I consider two-dimensional vortices (which find applications in three dimensions as models for magnetic flux tubes and cosmic strings) and three-dimensional magnetic monopoles. Sec. 4 discusses the application of Euclidean solutions to the treatment of tunneling phenomena as well as a brief description of sphalerons.

In a review of this size many aspects of the field must be left uncovered. Reviews containing a fuller discussion, including further references, of many of the topics covered here include several on magnetic monopoles (1-5) , nontopological solitons (6-8) , and instantons (9,10) , as well as some covering a broader range of topics (11-16) . A recently discovered family of solitons in Chern-Simons theories is reviewed in Ref.17.

2. THE KINK - A ONE-DIMENSIONAL EXAMPLE

A useful illustrative example is provided by a theory containing a single scalar field in one space and one time dimension. While containing many of the features encountered in more physically interesting cases, this model has the advantage that much of the analysis can be done explicitly. The theory is governed by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4}(\phi^2 - m^2/\lambda)^2 \tag{1}$$

There is a symmetry $\phi \to -\phi$, but this is spontaneously broken by the existence of two degenerate vacua at $\phi = \pm m/\sqrt{\lambda}$. In either of these vacua the elementary particles of the quantum theory have mass $\sqrt{2}\, m$.

The classical field equation following from this Lagrangian is

$$\ddot{\phi} - \phi'' = \lambda\phi\left(\phi^2 - \frac{m^2}{\lambda}\right) \tag{2}$$

where dots and primes refer to time and space derivatives, respectively. Of interest to us are solutions which remain localized in space. In principle, these could be static or they could have a periodic or quasiperiodic time dependence. In practice, however, it turns out to be rather difficult to find solutions with any but the simplest time-dependence. Restricting ourselves therefore to static solutions, we can drop the $\ddot{\phi}$ term and multiply both sides of Eq. (2) by ϕ'. The resulting equation can be rewritten as

$$\frac{d}{dx}\left[\frac{1}{2}\phi'^2 - \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2\right] = 0 \tag{3}$$

implying that the quantity in brackets must be an x-independent constant. Since any finite energy solution has $\phi = \pm m/\sqrt{\lambda}$ at spatial infinity, this constant must vanish. It follows that

$$\phi' = \pm\sqrt{\frac{\lambda}{2}}\left(\phi^2 - \frac{m^2}{\lambda}\right) \tag{4}$$

This can be integrated to give

$$\phi(x) = \pm\frac{m}{\sqrt{\lambda}}\tanh\left[\frac{m}{\sqrt{2}}(x-a)\right] \tag{5}$$

where a is a constant of integration.

Taking the upper choice of sign gives the so-called kink solution (18-20) . (The solution with the opposite sign may be termed an antikink.) The kink is rather localized, in that it deviates from a vacuum solution only within a region of width $\sim m^{-1}$ centered about the point $x = a$; the fact that it approaches different vacua to the left and to the right is irrelevant to any local observations, since the two vacua are indistinguishable. We may think of it as a kind of classical particle — a soliton — whose mass is given by the energy of the classical solution. Integration of the energy density

$$\mathcal{E}(x) = \frac{m^4}{2\lambda}\operatorname{sech}^4\left[\frac{m}{\sqrt{2}}(x-a)\right] \tag{6}$$

obtained from the Lagrangian (1) gives

$$M_{cl} = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} \tag{7}$$

The dependence on the coupling should be noted. The kink mass is of order m^2/λ times greater than that of the elementary excitation of the theory; as the coupling approaches zero, the kink mass becomes infinite. Similarly, the magnitude of the field in the kink solution (measured relative to either one of the two vacua) grows inversely with the strength of the coupling. Thus, the kink is an essentially nonperturbative phenomenon — it cannot be seen be studying small fluctuations about the vacuum in the weak coupling limit.

The discussion up to this point has been purely classical. The transition to the quantum theory can be investigated by a variety of techniques (18,20-26). The essential result that one obtains is that, at least for the case of weak coupling, these classical objects survive the quantization process relatively unscathed. Thus, not only does the quantum theory possess particle states corresponding to these solitons, but the classical solution gives a good first approximation to the properties of these particles. The underlying reason for this is that the Compton wavelength $1/M$ of the kink is much smaller than its classical size $1/m$. This means that, without having to give it a very high energy, the kink can be sufficiently localized that the quantum fluctuations in its position do not smear out the classical solution.

Furthermore, the quantum fluctuations of $\phi(x)$ are small enough that they do not obscure the classical field profile. This statement needs some explanation. In any quantum field theory, the fluctuations in the field at a given point are infinite. However, it is only averages of the field over finite volumes which are actually measurable. It turns out that the quantities $\phi_L(x)$ obtained by averaging the field over a region of width L centered about the point x have fluctuations of the order of $(\ln mL)^{-1/2}$. (In $d > 2$ space-time dimensions, with $L \lesssim m^{-1}$, the fluctuations would be of order $L^{-(d-2)/2}$.) It is therefore possible to choose L to be much smaller than the width of the kink and yet still have the magnitude of the fluctuations much less than the overall variation $2m/\sqrt{\lambda}$ of the classical solution.

In the weak coupling limit the quantum corrections to the kink mass can be calculated perturbatively. To a first approximation this can be done by treating the kink as a fixed stationary object which provides a classical background for the quantum field theory. This is done by writing the operator field $\phi(x)$ as

$$\phi(x) = \phi_{kink}(x) + \psi(x) \tag{8}$$

where $\phi_{kink}(x)$ is a c-number field given by Eq. (5) (with some fixed value of a) and $\psi(x)$ is an operator field. Substitution of this decomposition into the Hamiltonian gives

$$H = \frac{2\sqrt{2}\,m^3}{3\ \ \lambda} + \int dx \left\{ \frac{1}{2}\dot{\psi}^2 + \frac{1}{2}\psi'^2 + \frac{m^2}{2}\left[3\tanh^2\left(\frac{m}{\sqrt{2}}(x-a)\right) - 1\right]\psi^2 \right\} + \int dx \left\{ m\sqrt{\lambda}\tanh\left(\frac{m}{\sqrt{2}}(x-a)\right)\psi^3 + \frac{\lambda}{4}\psi^4 \right\} \tag{9}$$

The first term, independent of ψ, is just the classical kink mass. The terms cubic and quartic in ψ are suppressed by factors of $\sqrt{\lambda}$ and λ, respectively, and may be treated as perturbations. This leaves a quadratic Hamiltonian which can be rewritten as a sum of harmonic oscillator Hamiltonians by expanding $\psi(x)$ in terms of normal modes; i.e., the solutions of

$$\left\{ -\frac{d^2}{dx^2} + m^2\left[3\tanh^2\left(\frac{m}{\sqrt{2}}(x-a)\right) - 1\right]\right\}\psi_j = \omega_j^2\psi_j(x) \tag{10}$$

These modes can all be found explicitly. There are two normalizable modes, with frequencies 0 and $\sqrt{3/2}\,m$, and a continuum of non-normalizable modes beginning at $\omega = \sqrt{2}\,m$.

The ground state of this Hamiltonian, in which each of the normal modes is in its lowest state, corresponds to the kink. In addition to the classical kink energy, there is a contribution of the form $\frac{1}{2}\sum\omega_j$ from the zero point energies of the oscillators. This sum (or more properly, integral, because of the continuous part of the spectrum) turns out to be divergent. However, two additional effects need to be included. First, since the kink mass should be measured relative to the vacuum, we must subtract a similar sum containing the frequencies ω_j^{vac} of the normal modes about the classical vacuum. Second, there is a contribution from the counterterms needed to cancel the divergences of the quantum theory. The mass counterterm δm^2 arises at order λ in ordinary perturbation theory. However, because the kink field is itself of order $1/\sqrt{\lambda}$, this term also contributes to the lowest order quantum correction. Adding (with appropriate regularization) these three divergent terms gives

$$
\begin{aligned}
M_{kink} &= M_{cl} + \frac{1}{2}\sum_j \omega_j - \frac{1}{2}\sum_j \omega_j^{vac} + \int dx\, \delta m^2 \phi_{kink}^2(x) + O(\lambda) \\
&= \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} + c_0 + O(\lambda)
\end{aligned}
\tag{11}
$$

where c_0 is finite, of order unity, and calculable (18). The term of order λ has also been calculated (27) and the calculation can, at least in principle (28,29) , be continued to arbitrary order in λ.

The excited states of the Hamiltonian are obtained by giving nonzero occupation numbers to some of the normal modes. Physically, they correspond to a kink plus a number of elementary bosons, with the states of the latter determined by which of the normal modes are occupied. The continuum modes approach plane waves far from the kink, and correspond to unbound bosons scattering off the kink. The discrete mode with $\omega = \sqrt{3/2}\, m$ corresponds to a state with an elementary boson bound to the kink.

The zero frequency mode, however, is quite different. This mode arises from the fact that the kink breaks the translational symmetry of the Lagrangian. It is given explicitly by

$$
\begin{aligned}
\psi_0(x) &= N \frac{m^2}{\sqrt{2\lambda}} \operatorname{sech}^2\left[\frac{m}{\sqrt{2}}(x-a)\right] \\
&= N\, \frac{d\phi_{kink}(x)}{dx}
\end{aligned}
\tag{12}
$$

where N is a normalization constant. An infinitesimal deformation of the kink of the form $\phi_{kink} \to \phi_{kink} + \epsilon\psi_0$ is equivalent to a displacement of the kink to the right by an amount ϵN. The existence of the zero mode corresponds to the fact that such a displacement leaves the energy unchanged.

This mode must be treated on a different basis from the modes with nonzero frequencies since, after all, a zero frequency harmonic oscillator is not really an oscillator. This is done (23,28,30) by introducing a "collective coordinate" associated with translation of the kink. Essentially, this amounts to promoting the integration constant a to a time-dependent dynamical variable. The corresponding conjugate momentum P is determined by the way in which $\dot{a}$ enters the Lagrangian. Eq. (1) gives

$$\begin{aligned} L &= \frac{1}{2}\dot{a}^2 \int dx \left(\frac{d\phi_{kink}}{dx}\right)^2 + \cdots \\ &= \frac{1}{2} M_{cl}\, \dot{a}^2 + \cdots \end{aligned} \tag{13}$$

Here the dots represent terms which do not contain $\dot{a}$, as well as higher order effects arising from the possible deviation of $\phi(x)$ from $\phi_{kink}(x)$ due to the excitation of the nonzero-frequency modes. The integral on the first line is precisely equal to the classical kink mass (an explanation for this apparent coincidence will be given in the next section), so to lowest order $P = M_{cl}\, \dot{a}$, and the corresponding contribution to the Hamiltonian is $P^2/(2M_{cl})$. At higher orders in perturbation theory the M_{cl} in the denominator is replaced by the expansion (11) for the exact mass M_{kink} and the relativistic corrections to the kinetic energy begin to appear.

3. SOLITONS IN MORE THAN ONE SPATIAL DIMENSION

Solitons in more spatial dimensions present no issues of interpretation beyond those encountered in the study of the kink. The difficulty is in actually finding solutions. Even for static solutions the field equations are partial differential equations, and hence much harder to solve. Further, there is a result (31), known as Derrick's theorem, which forbids static solitons in scalar field theories with Lagrangians of the standard form. This result follows from the fact that a static solution of the field equations must be a stationary point of the energy functional

$$\begin{aligned} E[\phi(x)] &= \int d^n x \, \left[(\partial_i \phi)^2 + V(\phi)\right] \\ &\equiv T[\phi(x)] + U[\phi(x)] \end{aligned} \tag{14}$$

(Here $V(\phi)$ is understood to vanish at its minimum.) In particular, let us assume that $\phi(x)$ is such a solution, and consider the family of field configurations $\tilde{\phi}_\beta(x) = \phi(\beta x)$ obtained by rescaling its length scale. If the original configuration is indeed a solution, then the energy

$$E[\tilde{\phi}_\beta(x)] = \beta^{n-2} T[\phi(x)] + \beta^n U[\phi(x)] \tag{15}$$

must be stationary at $\beta = 1$. This implies that

$$T[\phi(x)] = \frac{n}{2-n} U[\phi(x)] \tag{16}$$

For $n = 1$ this gives $T = U = E/2$, thus explaining the apparently fortuitous equality, found in Eq. (13), between $P/\dot{a}$ and the kink mass. However, if $n \geq 2$ this relation cannot be satisfied with T and V both finite and positive, and hence the assumed solution cannot exist. The argument is easily extended to the case of many scalar fields.

To find solutions then, we must either introduce time-dependence or else go to theories with a more complicated structure. In either case the task of solving the field equations becomes more difficult. Rather than a brute force approach, one needs an understanding of the mechanisms which can give rise to particle-like solutions. Two broad classes of solutions, topological solitons and nontopological solitons, have been found. In both cases the existence and stability of the soliton can be traced to a conserved charge which it carries.

3.1 *Topological Solitons*

One class of solitons is based on the possibility of topologically nontrivial field configurations. An example of this is the kink solution studied in the previous section. The theory described by Eq. (1) has two degenerate vacua, $\phi = \pm m/\sqrt{\lambda} \equiv v$. In any configuration of finite energy, $\phi(x)$ must approach one or the other of these vacua as $x \to \pm\infty$. One can therefore divide all such configurations into four classes, with $(x(-\infty), x(\infty))$ equal to (v, v), $(-v, -v)$, $(-v, v)$, and $(v, -v)$. While any two configurations within the same class can be smoothly deformed one into the other, it is not possible to continuously go from one class to another without passing through configurations of infinite energy. Now recall that any local minimum of the energy functional gives a solution of the static field equations. It should therefore be possible to obtain static solutions by minimizing the energy within each of the four classes of configurations. The minima within the first two classes are just the two vacua. For the last two classes the minima clearly cannot be vacuum solutions, and so must be solitons; they are the kink and the antikink, respectively. Note the power of this argument: even if we had not been able to obtain an analytic expression for the kink solution, we would still be assured that a soliton did in fact exist.

In this theory there can be multisoliton configurations containing a number of kinks and antikinks, although these will not be static solutions. By annihilation of kink-antikink pairs, such a configuration can evolve into one containing either a single kink, a single antikink, or no solitons at all, according to whether the number of kinks less the number of antikinks is 1, -1, or 0. (Because kinks and antikinks necessarily alternate in position, these are the only possible values.) This can be stated more formally by defining a topological charge $Q = (\phi(\infty) - \phi(-\infty))/(2v)$, so that kinks and antikinks have $Q = 1$ and $Q = -1$, respectively. From the above remarks it is evident that Q is conserved, and that the topological charge of a multisoliton configuration is the sum of the charges of the individual solitons.

These ideas are readily extended to more spatial dimensions. To illustrate this, I first discuss below a two-dimensional example, the vortex solution, and then describe the three-dimensional magnetic monopole solutions. A somewhat different type of topological soliton, the skyrmion, is discussed elsewhere in this volume (32).

3.1.1 THE VORTEX — A SOLITON IN TWO DIMENSIONS A two-dimensional topological soliton (33) occurs in the Abelian Higgs model, in which a $U(1)$ gauge symmetry is broken by a complex Higgs field ϕ. Instead of the two degenerate minima of the theory of Eq. (1), the Higgs potential now has a continuous family of minima, given by $\phi = ve^{i\alpha}$. This is just what we need, since in two dimensions spatial infinity is not just the two points $x = \pm\infty$, but rather an infinite set of points which may be viewed as a circle at $r = \infty$. A configuration can be classified topologically by the behavior of ϕ around this circle. Thus, let $\phi(r = \infty, \theta) = ve^{i\alpha(\theta)}$. In order that the field be continuous, the net change in the phase α as θ varies from 0 to 2π must be equal to 2π times some integer n, which may be defined to be the topological charge of the configuration. This charge acquires further significance from the observation that for the energy to be finite the covariant derivative $D_i\phi = (\partial_i - ieA_i)\phi$ must fall sufficiently rapidly at large distance. Because the θ-dependence of ϕ implies that $\partial_i\phi$ falls only as $1/r$, there must be a nonzero vector potential $A_i = -(i/e)\partial_i(\ln\phi)$ at large r. Although this potential, being a pure gradient, gives a vanishing magnetic field F_{12} at large distance, the application of Stokes' theorem shows that the total magnetic flux $\Phi = \int d^2x\ F_{12}$ must be equal to $2\pi n/e$.

To obtain a soliton, we look for the configuration of minimum energy among those with unit topological charge; by appropriate choice of gauge, this configuration can be taken to have $\phi(r=\infty,\theta)=ve^{i\theta}$. In contrast with the case of the kink, this solution cannot be obtained analytically. Instead, the field equations must be solved numerically. One finds that the solution is centered about a point where $\phi(x)$ vanishes. (The existence of such a point is a consequence of the boundary conditions at spatial infinity.) In a region of radius $\sim 1/(ev)$ about this point A_i is not simply a gradient, and gives the required nonzero magnetic flux. This solution is known as the vortex.

By adding one more spatial dimension, z, and taking the fields to be z-independent, this two-dimensional particle-like solution can be turned into a three-dimensional string-like solution with magnetic flux flowing along the string. Recalling that the Abelian Higgs model in three space dimensions is essentially the same as the Landau-Ginzburg model for superconductivity, with the symmetric and asymmetric vacua corresponding to the normal and superconducting phases, respectively, we see that the vortex solution gives a model for magnetic flux tubes in a superconductor, with the quantization of the topological charge being equivalent to the quantization of the magnetic flux.

An anti-vortex solution, with topological charge -1, can be obtained from the vortex by the substitutions $\phi(x)\to\phi^*(x)$ and $A_i(x)\to -A_i(x)$. One can also seek solutions with $n>1$. Topological arguments alone cannot determine whether these exist, since the configuration which minimizes the energy could well be n widely separated unit vortices. It turns out that rotationally invariant solutions with topological charge $n>1$ do exist, but their stability depends on the details of the Higgs potential. In terms of the superconductivity analogy, stable higher charged vortices exist if the theory is Type I, but not if it is Type II.

In order to extend these arguments to other two-dimensional theories, and to motivate the generalization to the three-dimensional case, it is helpful to formulate them in more general terms.[2] Let us define two loops in a manifold M to be equivalent if one loop can be continuously deformed into the other. On a simply

2 For a fuller treatment of the topological methods used in this section, see Refs. (1,4,13,16).

connected manifold, where any loop can be continuously shrunk to a point, all loops are equivalent. If the manifold is not simply connected, there are a number of equivalence classes, which are the elements of the first homotopy group, $\Pi_1(M)$. Multiplication in this group corresponds to tracing out one loop after the other, while the identity element corresponds to the trivial loop containing a single point.

In a theory where a symmetry group G is spontaneously broken to a subgroup H the values of the scalar field which minimize the potential form a manifold M, which may be identified with the quotient group G/H. As one goes around a loop at spatial infinity, the fields $\phi(r = \infty, \theta)$ trace out a loop in M. The topological charge of the configuration can be identified with the corresponding element of $\Pi_1(M)$. Topologically stable solitons exist whenever $\Pi_1(M)$ is nontrivial.

Thus, in the Abelian Higgs model the manifold M is topologically equivalent to a circle, S^1. The fundamental group $\Pi_1(M) = \Pi_1(S^1) = Z$, the additive group of the integers. This reflects both the quantization of the topological charge and the fact that the topological charge of a multivortex configuration is the sum of the charges of its component vortices. Other possibilities arise in other gauge groups. For example, one can find theories where $\Pi_1(M) = Z_2$, the group formed by addition modulo two. In these theories topological charge is added modulo two, so that a two-vortex configuration has the same topological charge as the vacuum. Thus, two unit vortices can annihilate one another.

3.1.2 THE $SU(2)$ MAGNETIC MONOPOLE These ideas are readily generalized to three dimensions, where spatial infinity can be viewed as a two-sphere, S^2. Instead of tracing out a loop in M, the scalar fields at infinity map out a closed two-dimensional surface in M. Instead of $\Pi_1(M)$, the relevant quantity is now the second homotopy group, $\Pi_2(M)$, which is the group of equivalence classes of maps from S^2 to M. Topological solitons exist in theories in which $\Pi_2(M)$ is nontrivial.

The simplest example (19,34) of this occurs in an $SU(2)$ gauge theory with the symmetry spontaneously broken to $U(1)$ by a triplet Higgs field; I will describe this $U(1)$ with the language of electromagnetism. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}(D_\mu \phi)^2 - \frac{1}{4}F^4_{\mu\nu} - \frac{\lambda}{4}(\phi^2 - v^2)^2 \tag{17}$$

The elementary particles of the theory include a massless photon, two vectors with mass $m_V = ev$ and charges $\pm e$, and a neutral scalar with mass $m_S = \sqrt{2\lambda}\, v$.

For this theory the manifold M consists of the set of all isovectors with length v and is topologically equivalent to a two-sphere. Because $\Pi_2(S^2) = Z$, the theory should have solitons carrying an additive topological charge which, with a suitable normalization, can take on any integer value.

The solution with unit topological charge can be obtained by requiring that at large distances the Higgs field approach the "hedgehog" configuration $\phi^a(r) = vr^a/r \equiv v\hat{r}^a$ in which the direction of ϕ in the internal symmetry space is correlated with the angle in physical space. (Superscripts on fields are $SU(2)$ indices, while subscripts denote spatial components.) Because of the changing direction of the Higgs field, $\partial_i\phi$ falls only as $1/r$ at large distances. To have finite energy, the covariant derivative $D_i\phi$ must fall faster than this. As with the vortex, this is achieved by having a suitable long range vector potential. In this case, the appropriate choice is

$$A^a_i \sim \epsilon_{aij}\,\frac{\hat{r}^j}{er} \tag{18}$$

which gives a field strength

$$F^a_{ij} \sim \epsilon_{ijk}\,\hat{r}^a\hat{r}^k\frac{1}{er^2} \tag{19}$$

This is parallel to ϕ in internal space, and thus should be interpreted as purely electromagnetic. It is in fact the Coulomb magnetic field corresponding to a magnetic monopole with magnetic charge $Q_M = 1/e$. Similarly, one can show that any configuration with topological charge n carries magnetic charge $Q_M = n/e$. (However, there are no static solutions with multiple magnetic charge, except in the mathematically interesting, but unphysical, limit (35,36) of vanishing scalar meson mass.)

The full monopole solution can be obtained by multiplying the asymptotic forms of the fields by functions of r:

$$\phi^a = \hat{r}^a v h(r) \tag{20}$$

$$A_i^a = \epsilon_{aij}\, \hat{r}^j \, \frac{1 - u(r)}{er} \tag{21}$$

Substituting this ansatz into the field equations yield two coupled differential equations. The boundary conditions are that $h(\infty) = 1$ and $u(\infty) = 0$ (to agree with the presumed asymptotic behavior) and that $h(0) = 0$ and $u(0) = 1$ (so that the fields are nonsingular at the origin).

Although these equations can only be solved numerically, a rough qualitative picture yields some of the essential features of the solution. In this picture the monopole is viewed as having a core of radius R_{core} in which $h \neq 1$ and $u \neq 0$, with the fields having their asymptotic form for $r > R_{core}$. The mass of the monopole can then be divided into a contribution from the core and one from the Coulomb magnetic field outside the core. If the energy density inside the core is approximated by a constant ρ_0, this gives

$$M_{mon} \approx \frac{4\pi}{3}\rho_0 R_{core}^3 + \frac{2\pi}{e^2 R_{core}} \tag{22}$$

Minimizing with respect to R_{core} leads to $M_{mon} \approx 8\pi/(3e^2 R_{core})$. Now the only distance scales in the problem are the Compton wavelengths of the massive elementary particles. If these are roughly equal, we would expect $R_{core} \sim 1/m_V = 1/(ev)$, and hence $M_{mon} \approx 8\pi v/(3e)$. By comparison, numerical solution of the differential equations gives $M_{mon} = 4\pi v C/e$, where C ranges from 1 to 1.787 as λ/e^2 varies from 0 to ∞.

Further understanding of the structure of the monopole can be gained by applying a singular gauge transformation to make the orientation of ϕ uniform. The fields can then be written in the form

$$\phi^a = \delta^{a3} v h(r) \tag{23}$$

$$W_j \equiv A_j^1 + iA_j^2 = f_j(\theta,\phi)\,\frac{u(r)}{er} \tag{24}$$

$$A_j^3 = \epsilon_{ij3}\, r_i\, \frac{1-\cos\theta}{er^2\sin^2\theta} \tag{25}$$

In this gauge the unbroken $U(1)$ defined by ϕ corresponds to the 3-direction in internal space. The fields A_j^1 and A_j^2 correspond to the massive charged vector bosons of the theory; to emphasize this, they have been combined into a single complex vector field. The electromagnetic vector potential is A_j^3, which is singular along the negative z-axis; this singularity is precisely the Dirac string singularity of a $U(1)$ magnetic monopole (38). Within the $U(1)$ theory this string is unavoidable, but it is unobservable as long as $2q\,Q_M$ is an integer for any possible electric charge q; the remarkable effect of embedding the $U(1)$ in a larger gauge group is that the string can be eliminated completely.[3]

In this gauge it is easy to show that a new solution can be obtained by a phase rotation of the form $W_j(x) \to e^{i\alpha} W_j(x)$. This leads to a zero frequency mode when the quantum corrections to the monopole are calculated. As with the translational zero modes, this must be treated by introducing a collective coordinate, which in this case is a periodic variable specifying the overall phase of the solution (39-41). The momentum conjugate to this variable is proportional to the electric charge Q. Because the overall phase is a periodic variable, this conjugate momentum is quantized; a detailed calculation shows the unit of charge to be e. Just as the linear momentum gives a contribution $P^2/(2M)$ to the energy, the charge gives

3 Since all the charged elementary particles of the theory have unit electric charge, one might have expected a monopole with $Q_M = 1/(2e)$. However, we could have included in the model an isospinor field, whose particles would have charges $\pm e/2$; this would require that the minimum magnetic charge be $1/e$, which is indeed what we find.

an additional energy of the form $Q^2/(2I)$. Here I, which can be thought of as a moment of inertia in internal space, is obtained from a spatial integral of $|W_j|^2$; it is of the order of $1/(ev)$. Thus, built upon the monopole is a series of dyons carrying both electric and magnetic charge. Their masses, like that of the monopole, are of order v/e, while the mass splitting between successive dyons is of order e^3v.

Although the dyons appear here as time-dependent solutions, they can be made time-independent by an appropriate gauge transformation; it is in this form that they were first found.

3.1.3 MONOPOLES IN LARGER GAUGE GROUPS Monopole solutions can also occur in gauge theories with larger gauge groups and with a variety of scalar field representations. The only requirement is that the full gauge group G and the unbroken subgroup H be such that $\Pi_2(M) = \Pi_2(G/H)$ is nontrivial. In particular, it can be shown that if G is simple or semisimple then $\Pi_2(G/H) = \Pi_1(H)$, and hence that monopoles exist if H is not simply connected.[4] This applies in particular to any grand unified theory. By its very definition, such a theory has a simple gauge group G. To agree with experiment, the unbroken gauge group must be the $SU(3) \times U(1)$ of the strong and electromagnetic interactions; because of the $U(1)$ factor, this is not simply connected, and $\Pi_2(G/H) = \Pi_1(SU(3) \times U(1)) = Z$. Thus, any grand unified theory must contain magnetic monopoles.

The mass of these monopoles is determined by the symmetry-breaking scale at which a nontrivial Π_2 first appears. In the simple $SU(5)$ model an adjoint Higgs field ϕ with $\langle\phi\rangle = v_{GUT} \sim 10^{15}$ - 10^{16} GeV breaks the symmetry to $SU(3) \times SU(2) \times U(1)$. This is further broken to $SU(3) \times U(1)$ by a second scalar field χ, in the fundamental representation, with $\langle\phi\rangle = v_{EW} \sim 250$ GeV. The first level of symmetry breaking gives rise to monopoles (43) with unit magnetic charge $1/e$ and mass $\sim v_{GUT}/e$. Spherically symmetric solutions with two and three times the unit charge also exist, but are unstable (44) against dissociation into unit monopoles;

4 A technical point: this result assumes that G is simply-connected. This requirement can always be satisfied by taking G to be the covering group of the Lie algebra.

stable but less symmetric solutions with up to six units of magnetic charge (45) occur for certain ranges of parameters.

Other possibilities arise in more complicated models. For example (46,47), there are $SO(10)$ models where the symmetry is first broken to $SO(6) \times SO(4)$ by a scalar field ϕ_1 at a scale v_{GUT}. An explicit $U(1)$ factor appears only at a subsequent stage of symmetry breaking, when a second field ϕ_2 acquires a vacuum expectation value $v_2 \ll v_{GUT}$. The first symmetry breaking gives rise to monopoles of mass $\sim v_{GUT}/e$. Because $\Pi_2(SO(10)/SO(6) \times SO(4)) = Z_2$, these would have a Z_2 topological charge, with monopoles and anti-monopoles equivalent, if the $SO(6)\times SO(4)$ symmetry remained unbroken. When the $U(1)$ factor appears at the scale v_2, the homotopy group Π_2 is enlarged to Z, and the previous Z_2 monopoles acquire an ordinary magnetic charge of magnitude $1/e$; for these monopoles both ϕ_1 and ϕ_2 twist in a topologically nontrivial manner. However, a second type of monopole, in which only ϕ_2 twists, also occurs. These carry magnetic charge $2/e$ and have a much smaller mass $\sim v_2/e$.

Since magnetic monopoles are one of the definite predictions of grand unification, it is of considerable interest to know if any actually exist. Because of their great mass, they cannot be produced in any conceivable accelerator. However, the energies required were available in the very early universe. Indeed, fairly straightforward arguments (48,49) based on standard cosmology suggest that not only would monopoles have been produced at early times, but that enough would have survived to the present to far exceed the rather stringent upper bounds on the present-day monopole abundance. One of the motivations of the inflationary universe scenario (50) was to provide a solution to this primordial monopole problem. For a discussion of other approaches to the problem, and of the astrophysical and observational bounds on the monopole abundance, see Refs. 4 and 5.

3.1.4 ADDING FERMIONS — THE CALLAN-RUBAKOV EFFECT The coupling of fermions to a magnetic monopole leads to a number of unusual phenomena, including, e.g., the existence of objects with fractional fermion number (51). Perhaps the most important of these effects is the Callan-Rubakov effect (52-56) by which baryon number conservation is violated in the scattering of fermions by certain types of magnetic monopoles.

Angular momentum considerations give the first hint that scattering by a monopole might have unusual properties. A system containing both a particle with electric charge e and one with magnetic charge g has, in addition to any contributions from orbital motion or spin, an angular momentum of magnitude eg directed along the line from the electrically charged particle to the magnetically charged one. A classical electric charge moving directly toward the center of a magnetic monopole could not pass through to the other side, since this would require a sudden reversal of this angular momentum. The quantum mechanical analogue of such a radial trajectory is s-wave scattering off a monopole, and indeed, examination of the s-wave scattering states (57,58) reveals a mismatch between incoming and outgoing modes. This is seen, for example, in the solutions of the Dirac equation for a massless isodoublet fermion in the background of the $SU(2)$. The incoming solutions with vanishing total angular momentum and fermion number 1 are either left-handed with positive charge or right-handed with negative charge, while the outgoing states are either left-handed with negative charge or right-handed with positive charge; the charges are reversed for the modes with fermion number -1. It is evident, then, that one of the conserved quantum numbers of the fermion must change. A first guess might be that the fermion would change its electric charge, with the monopole compensating for this by becoming a dyon. However, the mass splitting between the dyon and monopole is too great to allow such charge transfer in low energy scattering. Instead, the issue is resolved by the effects (59,60) of the triangle anomaly (61,62) . Recall that the chiral current, whose conservation would appear to be guaranteed by the symmetry of the Lagrangian, acquires a nonzero divergence through one-loop quantum effects. For the case at hand this divergence is proportional to $\mathbf{E}\cdot\mathbf{B}$, the scalar product of the electric and magnetic fields. In the presence of the classical magnetic field of a monopole, the quantum fluctuations in the electric field can generate such a term and lead to a change in the total chiral charge. The net effect is that in the scattering of a massless fermion by the monopole there is a large amplitude for chirality nonconservation.

In more complicated theories analogous processes lead to nonconservation of other anomalous charges. In particular, violation of baryon number conservation can occur in grand unified theories. Of course, these theories have baryon number violation mediated by superheavy gauge bosons, even in the absence of monopoles. However, in ordinary low energy scattering of nucleons this violation is suppressed by a factor of $(E/M_{GUT})^4$ and is thus essentially unobservable. No such factor enters in the scattering of a nucleon by a monopole, and so the amplitude for baryon number violation in such scattering can be large (52-56). Although a precise calculation has not been performed, it is estimated that the cross-section for baryon number changing nucleon-monopole scattering is essentially geometrical in size, with $\sigma_{\Delta B} \sim 1/E^2$, where E is the energy of the nucleon. For a review of the subject, see Ref. 63.

3.2 *Nontopological Solitons*

A second type of soliton arises in theories where an unbroken symmetry gives rise to a conserved charge Q. These nontopological solitons are localized solutions with nonzero charge. Because this charge is of the same type as that carried by the elementary excitations of the theory, there is the possibility that it might be lost through emission of charged elementary particles. The stability of the soliton depends on whether or not such emission is energetically allowed and is therefore sensitive to the values of the parameters of the theory. In contrast with topological solitons, which in most cases occur only for a few low values of the topological charge, nontopological solitons (in three or more space dimensions) typically exist only if Q is greater than some minimum charge and often have no upper limit on their charge or mass. (The last property raises the possibility of solitons of truly astronomical size — soliton stars (64-68) .)

A wide variety of nontopological solitons have been found. (Some early examples are in Refs. 69-73.) Perhaps the simplest example (69,74) occurs in a theory involving a single complex scalar field ϕ in three space dimensions. The Lagrangian is of the form

$$\mathcal{L} = \frac{1}{2}|\partial_\mu \phi|^2 - V(|\phi|) \tag{26}$$

where V reaches its minimum at $\phi = 0$, so that the symmetry is unbroken; it is convenient to set $V(0) = 0$. This Lagrangian is invariant under the transformation $\phi(x) \to e^{i\alpha}\phi(x)$, with the corresponding conserved charge

$$Q = \int d^3x \, Im(\phi^* \dot{\phi}) \tag{27}$$

where the dot signifies a time derivative. At the classical level, this charge can take on any value. However, in the quantum theory Q is quantized (see the discussion of dyons above) and only takes on integer values.

If the solitons are to possess this charge, they clearly cannot be static. However, it is not hard to show that for fixed charge Q the solution which minimizes the energy has the quasistatic form

$$\phi(\mathbf{x}, t) = f(\mathbf{x})e^{i\omega t} \tag{28}$$

with $f = |\phi|$ real. For such solutions

$$Q = \omega \int d^3x \, |\phi|^2 \tag{29}$$

and so the energy can be written as

$$E = \int d^3x \left[\frac{1}{2}(|\nabla\phi|)^2 + V(|\phi|)\right] + \frac{Q^2}{2\int d^3x \, |\phi|^2} \tag{30}$$

For these quasistatic configurations the field equation reduces to

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = \frac{d}{d\phi}\left(V - \frac{1}{2}\omega^2|\phi|^2\right) \tag{31}$$

where the last term arises from $\ddot{\phi} = -\omega^2\phi$. This is just the equation for a static soliton in a theory with scalar field potential

$$\hat{V} \equiv V - (1/2)\omega^2|\phi|^2 \tag{32}$$

At large r the soliton must approach the vacuum solution $\phi = 0$. If it is to be stable, this value must be a minimum of $\hat{V}$, which implies that $\omega^2 < m^2 = (d^2V/d\phi^2)_{\phi=0}$, where m is the mass of the elementary charged particles of the theory. Furthermore, Derrick's theorem shows that such a soliton is possible only if $\hat{V}$ is negative for some values of ϕ, implying that $\hat{V}$ must have a second, deeper, minimum at some nonzero value of ϕ. In the region where $\hat{V}$ is negative we have $(2V/|\phi|^2) < \omega^2 < m^2 = (2V/|\phi|^2)_{\phi=0}$. Thus, the function $(2V/|\phi|^2)$ must achieve its minimum value ν^2 at some nonzero value of ϕ. It turns out (74,75) that the existence of such a minimum is essentially all that is needed for the existence of nontopological solitons in this model, and that solutions exist for all values of ω in the range $\nu < |\omega| < m$.

Matters become particularly simple in the limit of large Q. In this case the soliton turns out to be a sphere of radius R, inside of which $|\phi|$ has some constant value ϕ_0, surrounded by a surface region of thickness $\delta \sim m^{-1} \ll R$ in which ϕ goes to its vacuum value $\phi = 0$. There is a uniform charge density $\omega\phi_0^2$ in the interior of the soliton, giving a total charge

$$Q = \frac{4\pi}{3}\omega\phi_0^2 R^3 \tag{33}$$

which I will assume to be positive. R and ϕ_0 are determined by minimizing the energy (30) with Q held fixed. For sufficiently large R (and Q) the contribution to the energy from the surface region can be neglected relative to that from the interior and we can write

$$E = \frac{4\pi}{3}V(\phi_0)R^3 + \frac{3}{8\pi}\frac{Q^2}{\phi_0^2 R^3} + \cdots \tag{34}$$

where the dots represent terms which can be ignored in the limit of large R. Minimizing with respect to R, with ϕ_0 held fixed, gives

$$R = \left(\frac{3}{4\pi}\right)^{1/3} \left[2V(\phi_0)\phi_0^2\right]^{-1/6} Q^{1/3} \tag{35}$$

and hence

$$E = \sqrt{\frac{2V(\phi_0)}{\phi_0}}\, Q + \cdots \tag{36}$$

To minimize this ϕ_0 must be the value of the field at which $2V/|\phi^2|$ achieves its minimum value ν^2, and hence

$$E = \nu|Q| + \cdots \tag{37}$$

(The stability condition $\nu < m$ noted above can now be seen as the requirement that the mass to charge ratio of the soliton be less than that of the elementary particles of the theory.) From Eqs. (33) and (35) we now find that $\omega = \sqrt{2V(\phi_0)/\phi_0} = \nu$. Thus, in the large Q limit the fields in the bubble interior are independent of Q; solitons of this sort have been termed (74) Q-balls.

We can now go back and include the effects of the surface energy, which gives a contribution of the form ΣR^2 to Eq. (34). To leading order, this gives a correction of order $Q^{2/3}$ to the energy, and increases ω^2 above ν^2. A more detailed analysis reveals that ω increases as Q is decreased. Since ω is bounded from above by m, this leads to the existence of a minimum charge Q_{min} below which the soliton solution ceases to exist. Further, the soliton is stable only if its charge is greater than a value $Q_{stab} > Q_{min}$.

Nontopological solitons need not be Q-balls. It is possible to construct solutions whose energy, in contrast to that of a Q-ball, grows less than linearly with charge; e.g., in the theory of a charged scalar coupled to a neutral scalar with a broken discrete symmetry (76). A similar phenomenon occurs for bosonic field configurations which are stabilized by coupling to a fermionic field carrying a conserved charge; such objects (77-79) , which because of the presence of the fermionic field are not truly classical solutions, are related to the bag models (80,81) for hadrons. Finally, there are (82) Q-balls with a massless gauge field coupled to the soliton charge; in this case the Coulomb energy places an upper limit on Q.

4. EUCLIDEAN SOLUTIONS AND BARRIER PENETRATION

One can also find localized classical solutions in Euclidean space-time. These do not correspond to particles, as do their Minkowskian counterparts, but are instead related to quantum mechanical tunneling. The starting point for this connection is the WKB formula for the tunneling amplitude through a one-dimensional barrier:

$$\mathcal{A}_{WKB} \sim \exp\left[-\int_{x_1}^{x_2} dx\, \sqrt{2m(V(x)-E)}\right] \tag{38}$$

where the integral ranges over the entire classically forbidden region. To generalize this result to a system with more than one degree of freedom (83,84) , one considers all possible paths through the multidimensional barrier, calculates a tunneling probability for each path using the one-dimensional formula, and then maximizes this amplitude to find the most probable path. The leading approximation to the tunneling amplitude is given by the one-dimensional result for this path. Thus, for a system of N particles, all with mass m, with coordinates $q_1, q_2, \cdots, q_{3N}$, one must find the path $q_j(s)$ through configuration space which minimizes the integral

$$I = \int ds \left[\sum_j \left(\frac{dq_j}{ds}\right)^2\right]^{1/2} \sqrt{2m(V(q)-E)} \tag{39}$$

If the signs in front of $V(q)$ and E were reversed, this would be the principle of least action, which determines the trajectory of a classical mechanical system with fixed energy. But we know that this variational principle is equivalent to Hamilton's principle, which tells us to minimize the action $S = \int dt(T-V)$ and which leads to the Lagrangian equations of motion. The appropriate sign changes can be obtained by working with an imaginary time $t = ix_4$. Doing so, and then retracing the steps relating the two variational principles, one finds that a path which minimizes I is also a stationary point of the Euclidean action

$$S_E = \int dx_4 \left[\frac{1}{2m}\sum_j \left(\frac{dq_j}{dx_4}\right)^2 + V(q_j)\right] \tag{40}$$

and is given by a solution of the Euler-Lagrange equations in imaginary time. Furthermore, for this path $I = S_E$. The end points of this path are on the surfaces, on either side of the barrier, where $E = V$. Hence, at these end points the "velocities" dq_j/dx_4 vanish.

This method can be carried over to field theory (85-89) . The coordinates q_j are replaced by the values of the field at each point in space, $\phi(\mathbf{x})$, and the path $q_j(x_4)$ becomes a sequence of three-dimensional field configurations, $\phi(\mathbf{x}, x_4)$, which may itself be viewed as a field configuration in a four-dimensional Euclidean space. It should be stressed that x_4 is not a time in any physical sense, but rather simply a variable parameterizing a path through configuration space.

4.1 Vacuum Decay by Tunneling

One application of this method is to the decay of an unstable vacuum (85). An example of this arises in a theory with a scalar field governed by a potential $V(\phi)$ with two unequal minima, one a "false vacuum" at $\phi = \phi_f$ and the other a deeper "true vacuum" at $\phi = \phi_t$. The false vacuum state is stable classically, but quantum mechanically can decay via tunneling through the barrier in the potential energy

$$U = \int d^3x \left[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right] \tag{41}$$

This tunneling cannot go directly from the homogeneous false vacuum to a homogeneous state with $\phi \approx \phi_t$ because the volume integral makes the barrier between these infinite. Instead, the tunneling is to a state in which a bubble of true vacuum is embedded in a false vacuum background. A field configuration corresponding to such a bubble is shown in Fig. 1. Varying R while keeping the field profile in the wall region fixed gives a one-parameter family of configurations whose energy is plotted in Fig. 2. It is the sum of negative contribution $(4\pi/3)(\Delta V)R^3$, arising from the replacement of false vacuum by true in the interior, and a wall energy of the form $4\pi\sigma R^2$, due to both the gradient terms in the energy and the barrier in $V(\phi)$, which is traversed by the field as it passes through the bubble wall. These precisely cancel when $R = 3\sigma/(\Delta V)$, which would be the end point of the tunneling path if the system were constrained to this one-parameter set of configurations; at this point $\partial E/\partial R < 0$, indicating that the bubble would expand once it was nucleated. (In actuality, the bubble profile changes somewhat as one goes along the optimal path through the potential energy barrier.)

The rate for this process, as well as the optimal sequence of bubble profiles, can be obtained by solving the Euclidean field equations

$$\sum_{i=1}^{4}(\partial_i\phi)^2 = \frac{dV}{d\phi} \tag{42}$$

The boundary conditions are that $\phi(x) = \phi_f$ at the initial value of x_4^i, while the configuration at the final value of x_4^f is a bubble embedded in false vacuum. The interval between these is in fact semi-infinite; i.e., we must take $x_4^i = -\infty$, while x_4^f has a finite value which, by x_4-translation invariance, can be chosen to be 0. Because $\partial_4\phi$ vanishes at the end points of the tunneling path, the reflection $x_4 \rightarrow -x_4$ yields another solution, running from $x_4 = 0$ to $x_4 = \infty$, with the same action. Patching these two solutions together gives what is known as the bounce solution, whose Euclidean action $S_E = B$ is twice that of the original solution. This factor of two is the same as that arising when the tunneling amplitude is squared to obtain the tunneling probability, which is thus proportional to e^{-B}.

There are actually an infinite number of Euclidean solutions and tunneling paths, since the final bubble could equally well be located at any point in space. It is therefore more natural to speak of the probability per unit volume. This bubble nucleation rate per unit volume is of the form $\lambda = Ae^{-B}$. The prefactor A may be viewed as probing the energy barrier in directions orthogonal to the optimal tunneling path; if small deviations from this path have little effect on the one-dimensional tunneling amplitude, A is large, and conversely. An expression for A in terms of a functional determinant can be derived (90) by using path integral methods. However, in realistic applications it is seldom possible to evaluate this determinant, and one simply argues on dimensional grounds that $A \sim M^4$, where M is a typical mass scale of the theory.

4.2 Yang-Mills Instantons

Quantum mechanical tunneling can also arise within the context of a stable vacuum. The most important example of this is the Yang-Mills instanton (91), which describes a tunneling process in which both the initial and final field configurations are classical ground states.

For a Yang-Mills theory the classical energy is clearly minimized when the field strength $F_{\mu\nu}$ vanishes. (Here, and throughout this section, the field strength $F_{\mu\nu}$ and gauge potential A_μ should be understood to be matrices which are linear combinations of the generators of the gauge group.) This does not imply that the gauge potential also vanishes, but only that it must be gauge equivalent to zero, i.e., of the form $A_\mu = U^{-1}\partial_\mu U$, where $U(x)$ is an element of the gauge group. This degeneracy is greatly reduced when a gauge condition is imposed. In some gauges (e.g., axial gauge, $A_3 = 0$) one can impose conditions such that $A_\mu = 0$ is the unique classical ground state (13). However, in non-Abelian theories this can be done only at the (aesthetic) cost of allowing finite energy configurations for which the potentials do not vanish at spatial infinity. In other gauges (e.g., temporal gauge, $A_0 = 0$) these configurations are avoided, but an infinite set of degenerate classical ground states (92,93) remain. These correspond to gauge functions U_n which can be characterized by the integer winding number

$$n = \frac{1}{24\pi^2} \int d^3x \, \epsilon_{ijk} Tr \, U_n^{-1}\partial_i U_n \, U_n^{-1}\partial_j U_n \, U_n^{-1}\partial_k U_n \tag{43}$$

It is impossible classically to go from one of these degenerate ground states to the next (i.e., to change the winding number by one unit). However, quantum tunneling between these states is possible. This mixes the vacuum states of definite winding number, so that the true vacuum is a linear combination of these. The amplitude for this tunneling can be calculated by finding a solution of the Euclidean field equations such that the configurations at the initial and final values of x_4 (which turn out to be $\pm\infty$) are the two classical vacua; this solution is the instanton.

Tunneling also occurs in gauges with unique classical vacua (94). In these gauges the instanton describes a process in which the field starts at the vacuum and then tunnels through a potential energy barrier simply to get back to where it started. This is somewhat analogous to the case of a particle constrained to move on a vertically oriented ring, with the energy of the particle being too small to overcome the gravitational potential energy barrier at the top of the ring. Classically the particle will stay at the bottom of the ring, but quantum mechanically it can go around the ring by tunneling. Although its description is rather different in these two classes of gauges, the observable consequences of this instanton-induced tunneling are of course the same in all gauges (95,96) .

The instanton is in fact a four-dimensional topological soliton. As noted above, the configurations at $x_4 \to \pm\infty$ must be the desired initial and final configurations, which are of the form $U^{-1}\partial_\mu U$. Since the tunneling proceeds via finite energy configurations, the fields must also be of this form at spatial infinity. The instanton thus assigns an element $U(x)$ of the gauge group G to every point on the three-sphere at Euclidean infinity, $\mathbf{x}^2 + x_4^2 = \infty$. Any such assignment gives an element of the group $\Pi_3(G)$, which for any simple non-Abelian gauge group turns out to be the group of the integers.

Thus any such Euclidean configuration can be assigned an integer, called the Pontryagin index, given by

$$k = \int d^4x \left[\frac{1}{16\pi^2} TrF^{\mu\nu}\tilde{F}_{\mu\nu} \right] \tag{44}$$

where the dual field strength $\tilde{F}_{\mu\nu} = (1/2)\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}$. Although written here as a volume integral, the expression on the right hand side depends only on the fields at infinity, because the integrand can be written as the divergence of a current j_μ. Moreover, for vacuum solutions the spatial integral of j_0 reduces to Eq. (43)

for the winding number. Rewriting Eq. (44) as a surface integral and working in temporal gauge, it is easy to show that the contributions from the surfaces at spatial infinity vanish, so that k is equal to the difference of the winding numbers of the initial and final configurations. Tunneling between adjacent classical vacua is therefore described by a solution with unit Pontryagin number. By analogy with the arguments which ensure the existence of the vortex and monopole solutions, we know that minimizing the Euclidean action among the set of configurations with unit Pontryagin number will give such a solution.

In fact, an analytic expression for this solution can be found. The $SU(2)$ instanton (which is easily extended to larger gauge groups) may be written as

$$A_\mu(x) = \frac{-i}{g} \frac{(x-a)^2}{(x-a)^2+\lambda^2} U^{-1}(x-a)\partial_\mu U(x-a) \tag{45}$$

where $U(y) = (y_0 - i\mathbf{y}\cdot\sigma)/\sqrt{y^2}$. Its Euclidean action is $S_{instanton} = 8\pi^2/g^2$, where g is the gauge coupling. This solution depends on five real parameters. Four of these, the components of the vector a_μ, are a consequence of the translation invariance of the theory. The fifth parameter λ determines the size of the instanton. The freedom to choose λ arbitrarily reflects the scale invariance of the classical Yang-Mills theory and means that there are an infinite number of physically inequivalent tunneling paths. An instanton of size λ specifies a path through configuration space involving field configurations whose spatial extent is of the order of λ. Along this path the maximum potential energy (i.e., the maximum height of the barrier) is of the order of $1/(\lambda g^2)$. While this barrier height decreases as one goes to larger instantons, the length of the path through the barrier grows with λ in just such a manner that the tunneling action is unchanged.

When calculating the effects of tunneling, one must take into account the possibility of several succesive tunnelings, or of several roughly simultaneous tunneling processes taking place at widely separated points in space. This can be done by considering Euclidean configurations containing a "gas" of separated instantons and anti-instantons (97).[5] The higher action of these configurations can be outweighed by the increase in "entropy" arising from the freedom to choose the positions of the instantons and anti-instantons independently. Stated differently, the amplitude for tunneling via the path described by a configuration with two instantons and one anti-instanton is far smaller than that for tunneling by the path corresponding to a single instanton (a factor of $e^{-3S_{instanton}}$ compared to $e^{-S_{instanton}}$). However, because there are far more paths of the former type than of the latter (in a space of finite volume Ω, one is proportional to Ω^3 and the other to Ω), the former can dominate. If λ were fixed, the dominant contribution would be from configurations with a density of instantons in Euclidean space-time of the order of $\lambda^{-4}e^{-S_{instanton}}$. For weak coupling $S_{instanton}$ is large and we have a dilute gas in which the separation between instantons is large compared to their size. The problem is that λ is not fixed, and that we must therefore integrate over the sizes of the individual instantons. This leads to an integral which appears naively to diverge at both large and small λ. When one-loop quantum corrections (98-101) are taken into account, the g in the action becomes the running gauge coupling evaluated at a momentum of order $1/\lambda$. This insures the convergence of the integral as $\lambda \to 0$, but only makes the problem of calculating the effects of large instantons worse. This greatly complicates the task of obtaining reliable quantitative calculations of instanton effects in QCD.

On the other hand, the addition of a Higgs field, as in the electroweak theory, breaks the scale invariance at the classical level (98). This insures the convergence of the integral over λ, which is now dominated by the contributions from instantons of a single size.

4.3 Physical Consequences of Vacuum Tunneling

4.3.1 THETA PARAMETER AND CP VIOLATION The interference between instanton and non-instanton paths in the path integral can lead to CP violating effects.

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5 The anti-instanton, which has Pontryagin index -1, is obtained by making the substitution $(x-a)_4 \to -(x-a)_4$ in Eq. (45).

This is most easily seen by working in gauge with a unique classical vacuum, and recalling the analogy with the particle on a ring. In this analogue problem, suppose that a term of the form $(\alpha/2\pi)\dot{\theta}$ is added to the Lagrangian. Being a total time derivative, this term will not affect the classical equation of motion. However, in the quantum theory it introduces a relative phase factor of $e^{i\alpha}$ between the amplitudes for paths which tunnel through the gravitational potential energy barrier at the top of the ring and those which do not. Unless α is 0 or π, this gives parity violating effects even though the classical equations of motion are parity-invariant. In a similar fashion, the addition of the total divergence

$$\Delta\mathcal{L} = \frac{\theta}{16\pi^2} Tr F_{\mu\nu} \tilde{F}_{\mu\nu} \tag{46}$$

to the Yang-Mills Lagrangian density has no effect classically, but gives an extra phase factor $e^{i\theta}$ to trajectories which proceed by instanton-induced tunneling.[6]

If θ is neither 0 nor π, the interference effects from this additional term are both parity-violating and CP-violating. This might suggest that the inclusion of such a term in the QCD Lagrangian could provide an alternative explanation of the observed CP violation. However, it turns out that a θ large enough to account for the parity violation in the K meson system implies a neutron electric dipole moment well above the experimental upper limits. The essence of the difficulty is that the latter is now a purely strong interaction effect, while the former still involves weak interactions. By contrast, when CP-violation is attributed to phases in the Kobayashi-Maskawa matrix, both effects suffer the same weak interaction suppression.

To give an acceptably small neutron dipole moment, θ must be less than about 10^{-9}. It would seem that the simplest way to meet this constraint would be to set $\theta = 0$ and omit the entire term from the Lagrangian. However, when fermions

6 In gauges with multiple vacua, where the true vacuum is a linear combination of states with definite winding number, the effects of $\Delta\mathcal{L}$ can be mimicked by assigning appropriate phases to the expansion coefficients. The resulting states are called θ-vacua.

are included in the theory one finds that a chiral rotation of the fermion fields is equivalent to a shift in θ (92-93). A consequence is that the effective value of θ receives a contribution from the phases in the fermion mass matrix. These in turn depend on the phase of the scalar vacuum expectation value responsible for the mass generation. The result is that zero is not a particularly natural value for this final θ_{eff}. If there were at least one exactly massless quark, θ_{eff} could always be set to zero by a chiral rotation. Since this appears not to be the case, one must seek an explanation for the otherwise fortuitous fact that θ_{eff} is so small. One possible solution (102,103) to this "strong CP problem" is to add fields to the Lagrangian in such a way that a vanishing θ_{eff} is chosen dynamically. This gives an approximate new symmetry, whose breaking by instanton effects gives rise to the hypothetical axion (104,105) .

4.3.2 VIOLATION OF ANOMALOUS CONSERVATION LAWS Additional effects come into play if massless (or very light) fermions are present in the theory. In a fixed gauge field background the fermions can be described by finding the eigenfunctions of the Dirac equation and specifying which modes were occupied. If this background is varied, these eigenfunctions and their eigenvalues will change. If we regard the instanton-mediated tunneling process as the passage through a sequence of such gauge field configurations and follow the changes in the fermion eigenmodes over the course of the process, we find that the net effect is to shift the spectrum in such a manner that one left-handed mode is shifted from positive energy to negative energy, while one right-handed mode goes from negative to positive energy (97). If this process were sufficiently slow that the occupation numbers of the various eigenmodes did not change, the result would be the creation and annihilation of various particles. For example, if initially all the negative energy modes were filled and all the positive energy modes empty, then the final state would contain one unfilled left-handed negative energy state and one filled right-handed positive energy state. This would correspond to the creation of a right-handed antifermion and a right-handed fermion, thus changing the chiral charge by two units (92,93). The same change in chirality is found for other initial states; furthermore, this result can be shown to be exact (106) and not depend on the adiabatic approximation. (The underlying reason for this is that the integrand in Eq. (44) for the Pontryagin index is essentially the same as the anomalous divergence of the chiral current.)

Applying this result to QCD resolves the so-called $U(1)$ problem. It is well-known that the eight light pseudoscalar mesons can be interpreted as the Goldstone bosons of an approximate chiral $SU(3)$ symmetry which is spontaneously broken. However, the Lagrangian describing the coupling of three light quarks to the color gauge field has an approximate chiral $U(3)$ symmetry. This larger symmetry would lead one to expect a ninth light pseudoscalar meson; the absence of such a particle is the $U(1)$ problem. The issue is resolved once it is recognized that the instanton effects just described violate the extra symmetry, but not the chiral $SU(3)$.

A second application is to the electroweak theory. Because the baryon number current has an anomalous divergence, the shifts in the fermion energy levels due to $SU(2) \times U(1)$ instantons will lead to nonconservation of baryon number. However, the amplitude for tunneling is proportional to $e^{-S_{instanton}} \sim e^{-2\pi/\alpha_{weak}}$. Since the exponent is of the order of 200, the probability of this ever actually happening would appear to be negligible. (There is no enhancement from the integration over the instanton size, because this is fixed by the Higgs field.)

Recently, Ringwald (107) and Espinosa (108) have argued that in high energy scattering it may be possible to overcome this exponential suppression. The essential idea is that the tunneling induces an effective Lagrangian for baryon number violation which has interaction terms involving the product of the light fermion fields and arbitrary powers of the Higgs and gauge boson fields. These interactions are pointlike and thus lead to amplitudes for processes with many bosons in the final state which grow like powers of the energy. Moreover, if one calculates an inclusive quark-quark scattering cross-section, the sum over the number of final states bosons brings this energy-dependence into the exponent (109). This calculation suggests that baryon number violation becomes large at energies of the order of m_W/α_{weak}, and thus could be observable at the SSC. However, corrections to this result must be significant, since otherwise extrapolation to still higher energy would lead to violation of unitarity. At present the magnitude of these corrections remains unclear. For a recent review of the situation, see Ref. 110.

4.4 Thermal Fluctuations and Sphalerons

At finite temperature quantum tunneling through a potential energy barrier must compete with barrier crossing by means of thermal fluctuations. When the temperature is small compared to the height of the barrier, this proceeds primarily via paths which traverse the barrier near its lowest point. Associated with these paths is a Boltzmann factor $e^{-E_{s.p.}/T}$, where $E_{s.p.}$ is the energy of the saddle point configuration which lies at the high point on the lowest path across the barrier. Although other paths become important as the temperature approaches and then exceeds $E_{s.p.}$, knowledge of the saddle point is still the first step in the analysis of the problem.

For the decay of a metastable false vacuum, the saddle point configuration is one with a true vacuum bubble of critical size embedded within a false vacuum background. The radius of this critical bubble is such that the outward pressure from the true vacuum interior is just balanced by the inward push of the surface tension. It corresponds to the maximum of the curve of $E(R)$ in Fig. 2. (Actually, at high temperature one should use the free energy, rather than the energy, for determining the critical radius; qualitatively the picture is unchanged.)

Thermal fluctuations can also cross the barriers separating the degenerate classical vacua of Yang-Mills theory. For the case of an unbroken gauge symmetry the analysis of the problem is complicated by the fact that the scale invariance of the classical theory implies that there is no saddle point. Instead, the barrier height decreases monotonically as one goes along a direction in configuration space corresponding to field configurations of increasing spatial extent. (Roughly speaking, these correspond to cross-sections through instantons of increasing scale size.) While this would tend to favor fluctuations with larger spatial extent, the temperature provides an infrared cutoff and thus suppresses the largest configurations.

Matters are simpler for the case of a spontanously broken symmetry, where the Higgs field breaks the scale invariance. A saddle point now exists, and is known as the sphaleron. It is a solution of the static field equations but, because it is a saddle point, is unstable. In an $SU(2)$ theory with the symmetry broken by an isodoublet Higgs field ϕ with vacuum expectation value v, the sphaleron solution has the form (111,112)

$$A_i^a = \epsilon_{aij}\,\hat{r}^j\,\frac{f(r)}{r} \tag{47}$$

$$\phi = \frac{iv}{\sqrt{2}}\,h(r)\,\hat{r}^j\sigma^j\,\psi_0 \tag{48}$$

where the functions f and h vary from 0 to 1 as r ranges from 0 to ∞ and ψ_0 is a constant isospinor. (This solution was studied previously (113-115) in a different context.) The extension to the full $SU(2) \times U(1)$ electroweak theory can be obtained (116) by expanding about $\sin^2\theta_W = 0$. The sphaleron energy is then found to be a few times m_W/α_{weak}; i.e., of the order of 10 TeV. (To actually use the sphaleron to calculate high temperature barrier crossing, one must take into account the existence of a symmetry restoring phase transition and the thermal variation of the gauge boson mass. These and other effects are discussed in Ref. 117.)

Now recall from the previous section that in the electroweak model the vacuum tunneling described by the instanton leads to baryon number nonconservation. A similar violation of baryon number conservation occurs when thermal fluctuations carry the system over the potential energy barrier (118) and may have important consequences for the generation of the baryon asymmetry in the early universe. For a further discussion of these, see Ref. 119.

Acknowledgements: I thank Piet Hut, Kimyeong Lee, and Alfred Mueller for helpful comments on the manuscript. I would also like to acknowledge the hospitality of the Theoretical Physics and Theoretical Astrophysics Groups at Fermilab, where part of this review was written.

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FIGURE CAPTIONS

1) A field profile describing a true vacuum bubble of radius R in a false vacuum background.

2) The energy, as a function of the bubble radius, of the bubble depicted in Fig. 1.

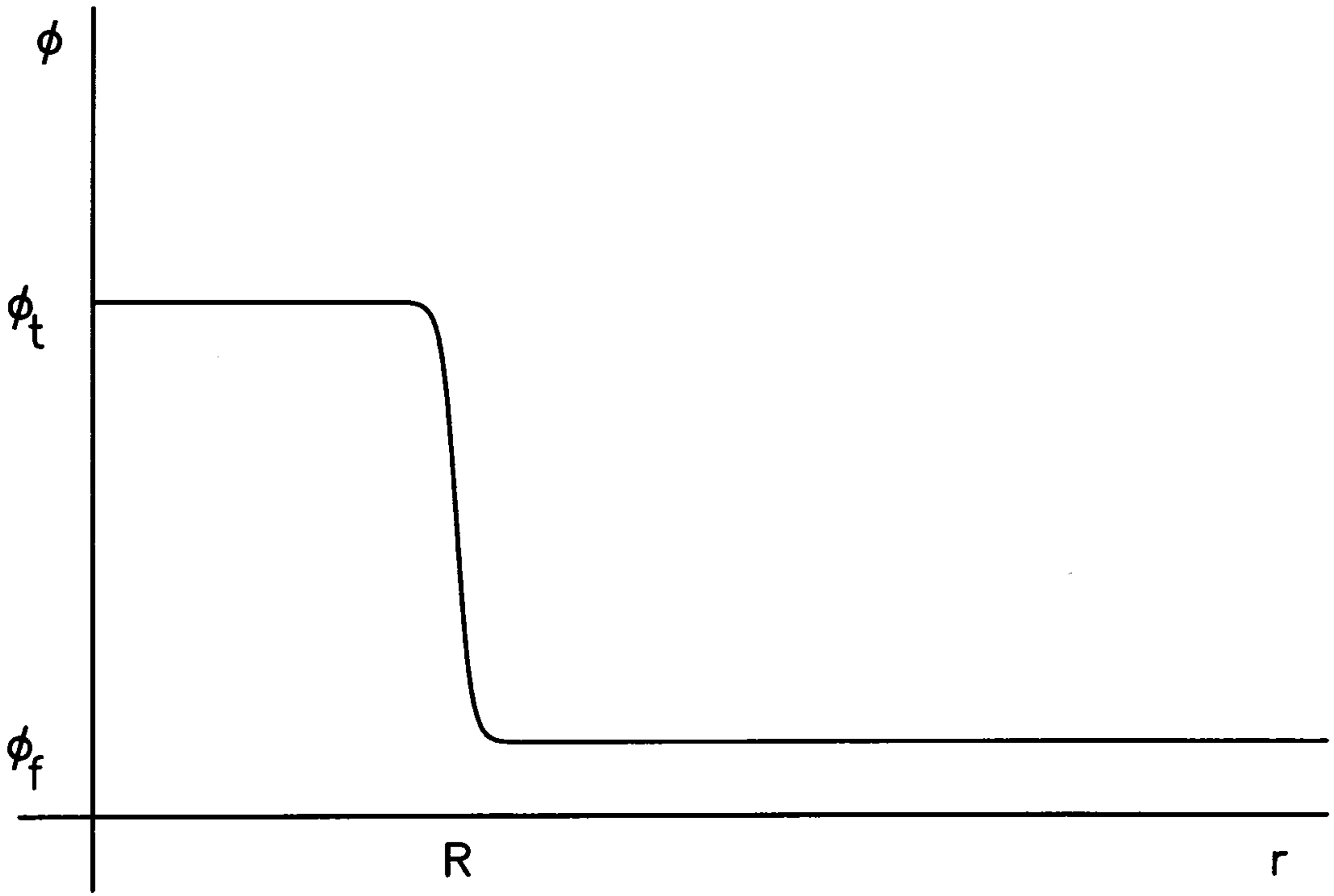

Figure 1

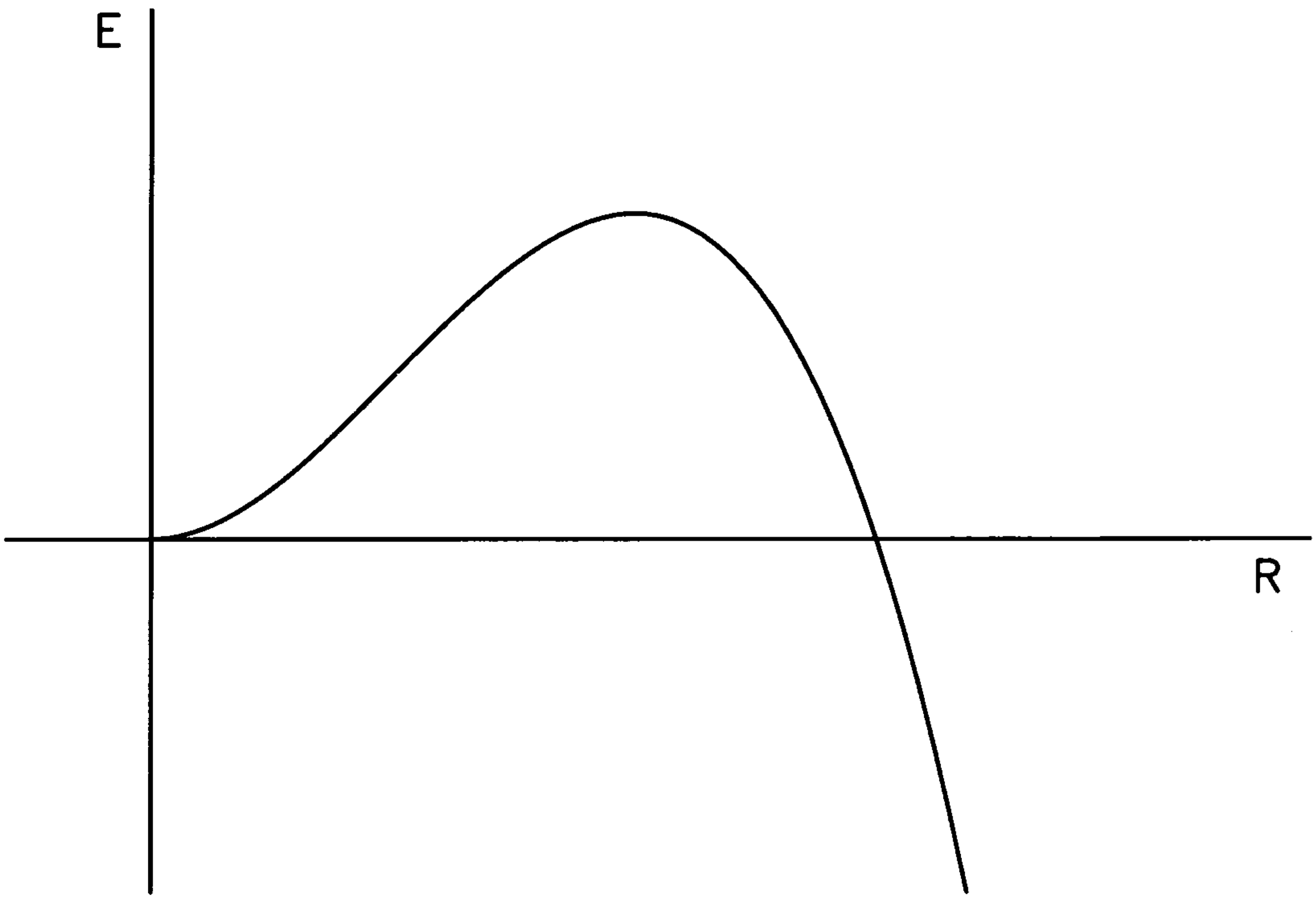

Figure 2